Quantum Field Theory II

Set 3: solutions

Exercise 1

- We compute the conserved momentum P_μ, recalling that the photon field can be written in Fourier space as:
$$A_\mu(x) = \int d\Omega_{\vec{k}} \left[a_\mu(\vec{k},t) + a^\dagger_\mu(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}.$$
From the Gupta-Bleuler Lagrangian we get $T^{0\nu} = -\partial^0 A^\rho \partial^\nu A_\rho + \frac{g^{0\nu}}{2}\partial^\alpha A^\rho \partial_\alpha A_\rho$. Let's first focus on the energy, which receives contribution from both terms:
$$\begin{aligned}
&P_0 = \frac{1}{2}\int d^3x(-\partial^0 A^\rho \partial^0 A_\rho + \partial^i A^\rho \partial_i A_\rho) = \\
&\int d\Omega_{\vec{q}}\, \frac{q^0}{4}\left[a_\rho(-\vec{q},t)a^\rho(\vec{q},t) - a_\rho(-\vec{q},t)a^{\rho\dagger}(-\vec{q},t) - a^\dagger_\rho(\vec{q},t)a^\rho(\vec{q},t) + a^\dagger_\rho(\vec{q},t)a^{\rho\dagger}(-\vec{q},t)\right] + \\
&\int d\Omega_{\vec{q}}\, \frac{q^i q_i}{4q^0}\left[a_\rho(-\vec{q},t)a^\rho(\vec{q},t) + a_\rho(-\vec{q},t)a^{\rho\dagger}(-\vec{q},t) + a^\dagger_\rho(\vec{q},t)a^\rho(\vec{q},t) + a^\dagger_\rho(\vec{q},t)a^{\rho\dagger}(-\vec{q},t)\right] = \\
&-\int d\Omega_{\vec{q}}\; q_0\; a^\dagger_\nu(\vec{q},t)a^\nu(\vec{q},t),
\end{aligned}$$
where we have used $q^i q_i = -q^i q^i = -q_0^2$ and dropped infinite constants as usual. For the spatial components, only the first term in the energy-momentum tensor gives a non vanishing contribution and we have:
$$\begin{aligned}
&P^i = \int d^3x(-\partial^0 A^\rho \partial^i A_\rho) = \\
&\int d\Omega_{\vec{q}}\, \frac{q^i}{2}\left[a_\rho(-\vec{q},t)a^\rho(\vec{q},t) + a_\rho(-\vec{q},t)a^{\rho\dagger}(-\vec{q},t) - a^\dagger_\rho(\vec{q},t)a^\rho(\vec{q},t) - a^\dagger_\rho(\vec{q},t)a^{\rho\dagger}(-\vec{q},t)\right] = \\
&-\int d\Omega_{\vec{q}}\; q^i\; a^\dagger_\nu(\vec{q},t)a^\nu(\vec{q},t),
\end{aligned}$$
where we have noticed that two of the four terms vanish by antisymmetry for $\vec{q} \to -\vec{q}$. Thus finally:
$$P_\mu = -\int d\Omega_{\vec{k}}\; k_\mu a^\dagger_\nu(\vec{k})a^\nu(\vec{k}).$$
Note that the sign is correct, since for the transverse components $a^\perp_\rho$ there is a $+$. Note also that P_μ is independent of time, so it is without loss of generality that we dropped the symbol t in last equation.

- When we deal with the quantization of massless vectors we have to define the physical states of the theory and in addition the physical observables. These can be defined as all the operators O that, applied to a physical state, still give a physical state. This translates into a condition involving the commutator $[O, L]$, where $L \equiv q^\rho a_\rho(q)$. Indeed:
$$\begin{aligned}
&L|\text{phys}\rangle = 0, \qquad\qquad O|\text{phys}\rangle = |s\rangle, \\
&L|s\rangle = LO|\text{phys}\rangle = [L, O]|\text{phys}\rangle + OL|\text{phys}\rangle = [L, O]|\text{phys}\rangle
\end{aligned}$$
If we want $|s\rangle$ to be a physical state we don't need to impose the vanishing of the commutator: is sufficient to require:
$$[L, O] \sim L\ .$$
We want to show that this is the case for the momentum $P_\nu = \int d^3x\ T_{0\nu}$, the Noether charge associated to translations. Since P_ν is a Noether charge we know that it can be regarded as the generator of translations, therefore we expect to find $[P_\mu, L] \sim \partial_\mu L$.

Computing explicitly the commutator using the previous result for the momentum we find,

$$[P_\mu, L(\vec{q})] = -\int d\Omega_{\vec{k}}\; k_\mu q^\rho \left[a^\dagger_\nu(\vec{k}),\, a_\rho(\vec{q})\right] a^\nu(\vec{k}) = -\int d^3k\, k_\mu q^\rho a_\rho(\vec{k})\delta^3(\vec{k}-\vec{q}) = -q_\mu q^\rho a_\rho(\vec{q}) = -q_\mu L(\vec{q}).$$

Then, defining $L(q) = L(\vec{q})e^{-iq^0 t}$, we get:

$$L(x) = \partial^\mu A^-_\mu(x) = \int d\Omega_{\vec{q}}\; e^{-iqx} L(\vec{q}) \implies [L(x), P_\nu] = i\partial_\nu L(x).$$

Exercise 2

The EOM is:

$$\Box A_\nu - (1-\xi)\partial_\nu\partial_\mu A^\mu = -J_\nu$$

which can be cast in the form:

$$\Pi_{\nu\mu}A^\mu(x) = -J_\nu(x)$$

where:

$$\Pi_{\mu\nu} \equiv (\Box\eta_{\mu\nu} - (1-\xi)\partial_\nu\partial_\mu)$$

One can solve formally for A_μ in terms of $(\Pi^{-1})^{\mu\nu}$:

$$A^\mu(x) = -(\Pi^{-1})^{\mu\nu}J_\nu(x).$$

At this point it is more convenient (but not compulsory) to work in momentum space:

$$\tilde{A}^\mu(k) = -\left(\tilde{\Pi}^{-1}\right)^{\mu\nu}\tilde{J}_\nu(x)$$

where:

$$\tilde{\Pi}_{\nu\mu} = k^2\left(\eta_{\nu\mu} - (1-\xi)\frac{k_\nu k_\mu}{k^2}\right).$$

In order to find $\left(\tilde{\Pi}^{-1}\right)^{\mu\nu}$, it is useful to notice that $\tilde{\Pi}_{\nu\mu}$ can be split into a sum of orthogonal projectors $P_L^{\mu\nu}$, $P_T^{\mu\nu}$:

$$P_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}\,, \qquad P_T^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

$$(P_T)^\mu{}_\alpha(P_T)^{\alpha\nu} = (P_T)^{\mu\nu}\,, \quad (P_L)^\mu{}_\alpha(P_L)^{\alpha\nu} = (P_L)^{\mu\nu}\,, \quad (P_L)^\mu{}_\alpha(P_T)^{\alpha\nu} = 0\,, \quad (P_L)^{\mu\nu} + (P_T)^{\mu\nu} = \eta^{\mu\nu}$$

$$\tilde{\Pi}_{\nu\mu} = k^2(P_T)_{\nu\mu} + k^2\xi(P_L)_{\nu\mu}.$$

Therefore, it is easy to check that $\left(\tilde{\Pi}^{-1}\right)^{\mu\nu}$ is simply given by the following expression:

$$\left(\tilde{\Pi}^{-1}\right)^{\mu\nu} = \frac{1}{k^2}(P_T)^{\mu\nu} + \frac{1}{k^2}\frac{1}{\xi}(P_L)^{\mu\nu} = \frac{1}{k^2}\left(\eta^{\mu\nu} + \frac{1-\xi}{\xi}k^\mu k^\nu\right).$$

For $\xi = 0$, this expression is not well defined. This is expected, because in the presence of gauge-invariance the longitudinal part of A^μ is not physical, therefore it cannot be determined from the EOM.

The choice $\xi = 1$ appears particularly simple:

$$\left(\tilde{\Pi}^{-1}\right)^{\mu\nu} = \frac{1}{k^2}\eta^{\mu\nu}$$

The Green function of the theory in momentum space is given by[1]

$$\tilde{G}^{\mu\nu}(k) = \eta^{\mu\nu}\frac{1}{(2\pi)^3}\frac{1}{k^2}$$

[1] The factor $(2\pi)^{-3}$ is conventional.

In order to find its expression in coordinate space, one has to take its anti Fourier-transform

$$G^{\mu\nu}(x) = \eta^{\mu\nu}\frac{1}{(2\pi)^3}\int d^4k e^{-ikx}\frac{1}{k^2} = \eta^{\mu\nu}\frac{1}{(2\pi)^3}\int dk^0\, d^3k\, e^{-ik_0t+i\vec{k}\vec{x}}\frac{1}{k_0^2-|\vec{k}|^2}$$

The results depend on the convention used to go around the poles at $k^0 = \pm|\vec{k}|$ in the complex k_0-plane.

Going above both poles corresponds to the *retarded* Green function $G_R^{\mu\nu}$. Indeed, for $t < 0$, the e^{-ik_0t} factor is exponentially suppressed for $\mathrm{Im}\, k_0 > 0$, therefore one can apply the Cauchy theorem for a closed integration contour contained in the upper complex plane. This does not embrace any pole, therefore the result of the integration is 0. Conversely, for $t > 0$, the closed contour goes in the lower half plane, and it contains both poles. By the Cauchy theorem, the result of the integration is:

$$G_R^{\mu\nu}(x) = \eta^{\mu\nu}\int dk^0 d^3k\, e^{-ik_0t+i\vec{k}\vec{x}}\frac{1}{k_0^2-|\vec{k}|^2} = \eta^{\mu\nu}\theta(t)\frac{1}{(2\pi)^3}\int d^3k\, 2\pi i\frac{1}{2|\vec{k}|}e^{i\vec{k}\vec{x}}\left(e^{i|\vec{k}|t}-e^{-i|\vec{k}|t}\right)$$

The integration in d^3k can be performed by going to polar coordinates as usual:

$$\begin{aligned} G_R^{\mu\nu}(x) &= \eta^{\mu\nu}\theta(t)\frac{i}{4\pi}\int_0^\infty d|\vec{k}|\int_{-1}^1 d\cos\theta\, |\vec{k}|e^{i|\vec{k}|r\cos\theta}\left(e^{i|\vec{k}|t}-e^{-i|\vec{k}|t}\right) = \\ &= \eta^{\mu\nu}\theta(t)\frac{1}{4\pi r}\int_0^\infty d|\vec{k}|\left(e^{i|\vec{k}|r}-e^{-i|\vec{k}|r}\right)\left(e^{i|\vec{k}|t}-e^{-i|\vec{k}|t}\right) = \\ &= -\eta^{\mu\nu}\theta(t)\frac{1}{8\pi r}\int_{-\infty}^\infty d|\vec{k}|\left(e^{i|\vec{k}|(t-r)}+e^{-i|\vec{k}|(t-r)}\right) \end{aligned}$$

Where in the last passage we omitted terms that would give rise to $\delta(t+r)$, which is 0 because $t+r>0$. The final result is:

$$G_R^{\mu\nu}(x) = -\eta^{\mu\nu}\theta(t)\frac{1}{4\pi r}\delta(t-r)$$

Instead, the Feynman propagation is defined by the prescription $k^2 \to k^2 + i\epsilon$ for $\epsilon \to 0^+$. This corresponds to going above the pole at $k^0 = |\vec{k}|$ and below the pole at $k^0 = -|\vec{k}|$. Indeed:

$$k_0^2 - |\vec{k}|^2 + i\epsilon \sim \begin{cases} (k_0+i\epsilon)^2 - |\vec{k}|^2\,, & k^0 > 0 \\ (k_0-i\epsilon)^2 - |\vec{k}|^2\,, & k^0 < 0 \end{cases}$$

Therefore, for $t > 0$ ($t < 0$) the closed contour embraces the pole at $k^0 = |\vec{k}|$ ($k^0 = -|\vec{k}|$). Thus:

$$G_F^{\mu\nu}(x) = -\eta^{\mu\nu}\frac{2\pi i}{(2\pi)^3}\theta(t)\int d^3k\frac{1}{2|\vec{k}|}e^{-i|\vec{k}|t+i\vec{k}\vec{x}} - \eta^{\mu\nu}\frac{2\pi i}{(2\pi)^3}\theta(-t)\int d^3k\frac{1}{2|\vec{k}|}e^{i|\vec{k}|t+i\vec{k}\vec{x}}$$

The Feynman prescription therefore implies the propagation of positive frequency in the future and negative frequencies in the past. It will be useful in the formalism of Feynman diagrams used in the perturbation theory for QFT. Performing the integration over angular variables one arrives at

$$G_F^{\mu\nu}(x) = \frac{-i\eta^{\mu\nu}}{2\pi r}\int_0^\infty d|\vec{k}|\sin(|\vec{k}|r)\left(\theta(t)e^{-i|\vec{k}|t}+\theta(-t)e^{i|\vec{k}|t}\right)\,.$$

Making use of the formulae,

$$\begin{aligned} \frac{1}{2i}\int_0^\infty\left[e^{ik|(r-t)}-e^{-ik(r+t)}\right] &= \lim_{\epsilon\to 0}\frac{r}{r^2-t^2+2i\epsilon t} = \frac{r}{r^2-t^2} - r\pi i\cdot sgn(t)\delta(t^2-r^2)\,, \\ \frac{1}{2i}\int_0^\infty\left[e^{ik(r+t)}-e^{-ik(r-t)}\right] &= \lim_{\epsilon\to 0}\frac{r}{r^2-t^2-2i\epsilon t} = \frac{r}{r^2-t^2} + r\pi i\cdot sgn(t)\delta(t^2-r^2)\,, \end{aligned}$$

one obtains the final expression for the Feynman propagator in the $\xi = 1$ gauge,

$$G_F^{\mu\nu}(x) = \frac{i\eta^{\mu\nu}}{2\pi(t^2-r^2)} - \frac{\eta^{\mu\nu}}{2}\delta(t^2-r^2)\,.$$

Exercise 3 (optional exercise)

- We consider a completely anti-symmetric field $F_{\mu\nu\rho}$ and want to decompose it into a direct sum of irreducible representations of the Lorentz group. We will do that in two ways.

  The most straightforward way consist in decomposing the tensor into the product of spinor representations of the Lorentz group, the $\left(\frac{1}{2},0\right)$ and the $\left(0,\frac{1}{2}\right)$[2]. We recall that the vector representation (V_μ) of Lorentz is the $\left(\frac{1}{2},\frac{1}{2}\right)$. We can therefore rewrite it in tensor notation as $V_{\dot{a}a}$, where $\dot{a}$ transforms under the left and a transforms under the right spinorial representation.

  We can now do the same with our tensor:

$$F_{\mu\nu\rho} \to F_{\dot{a}a\dot{b}b\dot{c}c} \tag{1}$$

  where we assume antisymmetry under the simultaneous exchange of $\dot{a}a$ and $\dot{b}b$ or $\dot{c}c$.

  Decomposing the tensor in irreducible representations is now simply a matter of decomposing the tensor into symmetric and antisymmetric part for both the dotted and undotted indices. This is done using the invariant antisymmetric tensors ϵ_{ab} and $\epsilon_{\dot{a}\dot{b}}$. It is simple to show that only one combination respects the antisymmetric properties of $F_{\mu\nu\rho}$, namely

$$F_{\dot{a}a\dot{b}b\dot{c}c} = \delta_{ab}\epsilon_{\dot{a}\dot{b}}\tilde{F}_{\dot{c}c} - \delta_{ac}\epsilon_{\dot{a}\dot{c}}\tilde{F}_{\dot{b}b} + \delta_{bc}\epsilon_{\dot{b}\dot{c}}\tilde{F}_{\dot{a}a} \tag{2}$$

  Therefore the anti-symmetric field $F_{\mu\nu\rho}$ transforms under the irreducible $\left(\frac{1}{2},\frac{1}{2}\right)$ representation.

  This result could habe been found more directly by using only vector indices and writting the tensor in terms of its dual $\tilde{F}_\mu$, which transforms under the $\left(\frac{1}{2},\frac{1}{2}\right)$:

$$F_{\mu\nu\rho} = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\tilde{F}^\sigma \tag{3}$$

- The equations of motion are

$$\partial_\mu F^{\mu\nu\rho} = 0 \tag{4}$$

- In complete analogy with the standard maxwell case, the antisymmetry of F implies a trivial conservation law:

$$\partial^\mu \epsilon_{\mu\nu\rho\sigma} F^{\nu\rho\sigma} = 0, \quad \text{or equivalently } \partial_\mu \tilde{F}^\mu = 0 \tag{5}$$

  This is the analogue of the homogeneous Maxwell equations for a 2-form field. In Maxwell's electrodynamics without matter, the theory is completely equivalent under exchange of the electric field and the magnetic field. This is the duality between the standard picture where we define $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ and the magnetic picture where we define $\tilde{F}_{\mu\nu} = \partial_{[\mu}\tilde{A}_{\nu]}$. Going from one picture to the other implies exchanging the equations of motions and the Bianchi identities. The equivalence is broken when adding matter which enables to distinguish between electric and magnetic charges. In the case of the 2-form, the two pictures are:

$$F_{\mu\nu\rho}(A) = \partial_{[\mu}A_{\nu\rho]}, \quad \text{and } \tilde{F}_\mu = \partial_\mu\tilde{\phi} \tag{6}$$

  This expression for $\tilde{F}$ is imposed by the equations of motion. In this case, the dual gauge field is simply a scalar field. In the next question we will see that this is not incompatible with the fact that they describe the same system.

- We can now determine the degrees of freedom. In the standard Maxwell case, even though the gauge field A_μ has four components, we only have two dynamical degrees of freedom. One of the components of A_μ is fixed by Gauss law $\partial_i F^{i0} = 0$, which is a first order differential equation and therefore only a constraint. Finally one component of A_μ is unphysical/redundant because of the gauge invariance of the theory.

  On the other hand, the 2-form gauge field $A_{\mu\nu}$ is antisymmetric and has therefore six independent components. We however have 3 Gauss laws $\partial_i F^{ij0} = 0$, where $j = 1, 2$ or 3, reducing the number of degrees of freedom to 3, by fixing the values of A_{0j}. Moreover, we lose 2 additional degrees of freedom from the gauge part of the field

$$A_{\mu\nu} \to A_{\mu\nu} + 2\partial_{[\mu}\omega_{\nu]}. \tag{7}$$

[2] Recall that the double-cover of the Lorentz group is given by $SL(2,\mathbb{C})$. Therefore the fundamental representation of $SL(2,\mathbb{C})$, namely the $\left(\frac{1}{2},0\right)$ is a representation of Lorentz.

One needs to be careful here. Because of the constraints from the Gauss law, ω_0 is already fixed and does not reduce the degrees of freedom. Moreover the longitudinal component of ω, does not contribute. Indeed, if $\omega_\mu(x) = \partial_\mu \alpha(x)$, for any continuous function α, $A_{\mu\nu}$ is not modified. Therefore imposing a gauge only further reduces the number of degrees of freedom to one. This is in agreement with the dual picture where the field strength is a vector and therefore the gauge field is only a scalar. We see here that even though in one case the theory is described using a 2-form $A_{\mu\nu}$ and in the other it is described by a single scalar $\tilde{\phi}$, the number of dynamical fields is the same, and the theories describe the same physics.

- We can rewrite the equations of motion and the Bianchi identities respectively as:

$$\partial_\mu \epsilon^{\mu\nu\rho\sigma} \tilde{F}_\sigma = 0, \quad \partial_\mu \tilde{F}^\mu = 0 \tag{8}$$

The equations of motion are now trivial equations following from the definition of $\tilde{F}$, while the Bianchi identity becomes the equation of motion for the field ϕ. Indeed the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{2}\tilde{F}^\mu(\phi)\tilde{F}_\mu(\phi) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \tag{9}$$

We recognize here that the theory is actually simply the theory of a free massless scalar field.

Formalities on the use of ∇^{-2}

Consider as an example the Coulomb condition $\vec{\nabla}\cdot\vec{A} = 0$, and the definition of the electric field $\vec{E} = -\vec{\nabla}A^0 - \partial_t\vec{A}$. Gauss law $\vec{\nabla}\cdot\vec{E} = \rho$ impiles $-\nabla^2 A^0 - \partial_t(\vec{\nabla}\cdot\vec{A}) = -\nabla^2 A^0 = \rho$ in Coulomb gauge. The formal solution to this equation is $A^0 = -\nabla^{-2}\rho$, so by explicitly finding A^0 we will become more familiar with the meaning and the implications of the operator ∇^{-2} on the right hand side.
Let's first introduce the Green function $G(\vec{x})$ for the Laplace operator, defined implicitly by the equation

$$\nabla^2_{(x)} G(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y}),$$

where the subscript (x) means that derivatives are taken with respect to $\vec{x}$. Once the Green function is known, the equation for A^0 is simply solved by expressing this field as a convolution of the Green function with the charge density, namely

$$A^0(\vec{x}) = -\int d^3y\; G(\vec{x}-\vec{y})\rho(\vec{y}) \equiv -[G*\rho](\vec{x}),$$

as it can be easily proved by applying the laplacian to both sides. The Green function for the Laplace operator in three (spatial) dimensions is

$$G(\vec{x}-\vec{y}) = -\frac{1}{4\pi|\vec{x}-\vec{y}|},$$

as it can be proved again by applying the laplacian to both sides (recall that $\nabla^2_{(x)} = \nabla^2_{(x-y)}$: for $\vec{x} \neq \vec{y}$ one can explicitly compute $\nabla^2_{(x)}\frac{1}{|\vec{x}|}$ getting zero, since $\frac{1}{|\vec{x}|}$ is the harmonic function in 3 spatial dimensions; for $\vec{x} = \vec{y}$ it is convenient to pass to Fourier space, and the factor $-1/(4\pi)$ comes out in a straightforward way).
Thus the solution for A^0 is

$$A^0(\vec{x}) = \int d^3y\; \frac{\rho(\vec{y})}{4\pi|\vec{x}-\vec{y}|},$$

which is the well-known definition of the electrostatic potential in classical physics. Notice that the presence of a negative power of ∇ introduces a non-locality in the solution, meaning that the field A^0, evaluated at $\vec{x}$, receives contribution from *every point* $\vec{y} \neq \vec{x}$ in which the charge density is present. Conversely, if the formal solution A^0 had involved only positive powers of derivative operators, no non-locality would have arisen.
Let's now express the operator ∇^{-2} in Fourier space. This space is well suited for solving the equation for A^0, since the transform ($\mathcal{F}$) of any convolution product is an algebraic product of transforms:

$$\begin{aligned}\mathcal{F}[A^0] \equiv \tilde{A}^0(\vec{p}) \equiv \int d^3x\; e^{-i\vec{p}\cdot\vec{x}}A^0(\vec{x}) = -\mathcal{F}[G*\rho] = -\int d^3x\; d^3y\; e^{-i\vec{p}\cdot\vec{x}}G(\vec{x}-\vec{y})\rho(\vec{y}) \\ = -\int d^3w e^{-i\vec{p}\cdot\vec{w}}G(\vec{w})\int d^3y e^{-i\vec{p}\cdot\vec{y}}\rho(\vec{y}) = -\tilde{G}(\vec{p})\tilde{\rho}(\vec{p}),\end{aligned}$$

where we have renamed $\vec{w} \equiv \vec{x} - \vec{y}$. Since the Fourier transform of $G(\vec{w}) \equiv -\frac{1}{4\pi|\vec{w}|}$ is $\tilde{G}(\vec{p}) = -\frac{1}{|\vec{p}|^2}$, then the equation for A^0, read in Fourier space, becomes $\tilde{A}^0(\vec{p}) = |\vec{p}|^{-2}\tilde{\rho}(\vec{p})$, and the following the general identity holds:

$$\mathcal{F}[\nabla^{-2} f] \equiv \int d^3x \; e^{-i\vec{p}\cdot\vec{x}} \nabla^{-2} f(\vec{x}) = -|\vec{p}|^{-2} \tilde{f}(\vec{p}). \tag{10}$$

One more comment on locality: the inverse Fourier transform of the function $|\vec{p}|^2$, namely $\int \frac{d^3p}{(2\pi)^3} |\vec{p}|^2 e^{i\vec{p}\cdot\vec{x}}$ is read in coordinate space as the distribution $-\nabla^2\delta^3(\vec{x})$, and similarly for any positive power (and, in general, any polynomial) of the gradient operator. Such a polynomial, convoluted with any test function, gives a sum (with finitely many terms) of derivatives of the test function evaluated in $\vec{x} = 0$, which states the locality of the expression. Conversely, functions that are written in momentum space as *infinite* Taylor series of p, or that are non-analytical in p, give rise in general to non trivial convolutions, i.e. to integrals that receive contributions from different points of the three-space, as we have seen in our electrostatic example.

As an application of identity (10), one can consider the operator $P^{\perp}_{ij} \equiv (\delta_{ij} - \partial_i\partial_j\nabla^{-2})$ that projects a generic vector field $V^i(\vec{x})$ onto its divergenceless part $V^i_{\perp}(\vec{x}) \equiv P^{\perp}_{ij} V^j(\vec{x})$, satisfying $\vec{\nabla} \cdot \vec{V}_{\perp}(\vec{x}) = 0$.
Recall first that the Fourier transform of the function $e^{i\vec{q}\cdot\vec{x}}$ is

$$\mathcal{F}[\exp] \equiv \int d^3x \; e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$

Using now (10), one has

$$\nabla^{-2} e^{i\vec{q}\cdot\vec{x}} = \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \mathcal{F}[\nabla^{-2}\exp] = \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} (-|\vec{p}|^{-2})(2\pi)^3\delta^3(\vec{p} - \vec{q}) = -|\vec{q}|^{-2} e^{i\vec{p}\cdot\vec{x}},$$

thus

$$V^i_{\perp}(\vec{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \tilde{V}^i_{\perp}(\vec{p}) = \int \frac{d^3p}{(2\pi)^3} \; \tilde{V}^j(\vec{p})(\delta_{ij} - \partial_i\partial_j\nabla^{-2}) e^{i\vec{p}\cdot\vec{x}} = \int \frac{d^3p}{(2\pi)^3} \; \tilde{V}^j(\vec{p})(\delta_{ij} - p_i p_j |\vec{p}|^{-2}) e^{i\vec{p}\cdot\vec{x}}$$

and the projector in Fourier space is then

$$\begin{aligned} \tilde{P}^{\perp}_{ij} &\equiv \delta_{ij} - \frac{p^i p^j}{|\vec{p}|^2}, \\ \tilde{V}^i_{\perp}(\vec{p}) &= P^{\perp}_{ij} \tilde{V}^j(\vec{p}). \end{aligned}$$

This has now an evident interpretation: given a vectorial operator $\tilde{\vec{V}}(\vec{p})$, function of some momentum $\vec{p}$, $\tilde{P}^{\perp}_{ij}$ projects the vector on the subspace transverse w.r.t the direction of motion, $\vec{p} \cdot \tilde{\vec{V}}^i_{\perp}(\vec{p}) = 0$.